Pensions By Dickens

(T.J. Fairclough C.Math., MIMA.)

Wickfield & Heep

The clock was striking the appointed hour on a drizzly December morning as David Copperfield tapped lightly on the counting-house window. Above his head and below the three brass balls a painted wooden sign proclaimed " Wickfield & Heep - Independent Financial Advisers". Inside Uriah Heep jumped down from his stool and thrust aside the great fat book in which his lank forefinger had been tracing each line as he read. Hurrying over to the door he unctuously admitted his visitor with a grinding of palms and his customary two bows and a scrape.

'Good morning Master Copperfield my dear Sir and most welcome I'm sure to my er... I should say of course *our* umble establishment. It is indeed the mark of a gentleman to be so punctual especially in such inclement weather if I may take the liberty of saying.

'Good Morning Heep, Mr. Micawber', said Copperfield, nodding to each as he entered the chamber, removing his stove-pipe hat to shake the raindrops from it and sitting down as he spoke. 'I have an appointment with Mr. Wickfield, please be so good as to tell him that I've arrived.' 'Bless you sir' replied Heep 'But unfortunately Mr. Wickfield is indisposed today and has asked me to stand in for him unless of course you have any objections Master Copperfield? - I should say Mister, but the other comes so natural,' said Uriah

'I suppose not' replied David unenthusiastically. 'Very well then let's get down to business straight away shall we? I've requested this meeting because I want some financial advice. Thanks to the generosity shown to me in the will of a distant relative, the late Mrs. Lirriper, I have recently inherited a modest lump sum. It is my present intention to invest this unexpected benefaction at 4% per annum with the Bank of England in a quantity sufficient to provide me a regular pension of forty pounds a year. Such an income, although considerably more than the seven shillings that was once my weekly stipend from Murdstone and Grinby, may yet be insufficient to meet all my future needs and I am interested to discuss ways of increasing the income. I'm willing to consider any prudent suggestions you may care to recommend.'

Mr. Micawber's Opinion

Mr. Micawber, sitting on the high stool in his capacity as confidential clerk to Uriah Heep, had been listening to the proceedings with interest. He now jumped down to say 'My dear Copperfield, I'm delighted to hear of your windfall and please accept my sincere commiseration on the sad circumstances that gave rise to it. With regard to such matters as income and expenditure I hold resolute opinions and my advice is easily stated[1] "*Annual income forty pounds, annual expenditure thirty-nine pounds nineteen and six, result happiness. Annual income forty pounds, annual expenditure forty pounds ought and six, result misery. The blossom is blighted, the leaf is* *withered, the god of day goes down upon the dreary scene, and - and in short you are forever floored. As I am*!"

Copperfield weighed these words of wisdom carefully before venturing to remark 'That's very clear Mr. Micawber, I will definitely bear those sensible precepts in mind and thank you most kindly. What do you say Heep?'

'Of course Master Copperfield, most excellent advice it is if I may be allowed to say. Particularly suitable advice it is on the *expenditure* side of the equation certainly but perhaps a trifle more guidance could yet be supplied on the *income* side of the equation sir if I may make so bold? Why, any fortunate gentleman blessed with such a considerable degree of capital might well increase the amount of income he draws from it. Of course the precise amount of income taken would depend on the particular circumstances of his age and health and his desire or otherwise to leave a fortune behind him when the time eventually comes for him to become a partaker of glory.'

'A trifle more guidance to be supplied Sir?' spluttered Mr. Micawber at last and with some asperity. 'Pray explicate to the present company on the specific shortcomings of my advice that you so evidently perceive and to which you have so kindly alluded!'

Uriah Heep's Advice

' Goodness gracious me Micawber', said Uriah Heep, 'A thousand apologies if my clumsy manner of speaking has led you to mistakenly believe I thought there are any shortcomings in your clear assessment of the situation I'm sure.'

Mr. Micawber, seemingly mollified, went on to briskly say 'Perhaps then Mr. Heep you would be good enough to explain how it is possible to take more than 4% per annum from an investment paying 4% per annum without, in short sir, ending up in the workhouse!?'

'By all means sir, if you will kindly allow me the opportunity I will attempt to clarify my meaning.' said Heep, hurriedly continuing. 'It is clear that the lump sum in question must be £1000 for that is the amount required to furnish £40 yearly interest when invested at 4% per annum. Obviously if interest rates remain fixed at 4% per annum and if only the £40 interest is spent each year as income then the £1000 capital will of course survive intact, continuing to earn £40 a year for ever and eventually pass into the hands of the estate beneficiaries. Indeed for this reason such an income producing investment may be termed a *perpetuity*.'

'If Master Copperfield wishes however, he could increase the annual income over and above £40 by simply taking out instead say £50 or even £100 a year. Of course taking the extra income would eat into the original £1000 capital. The capital could even disappear altogether if Master Copperfield outlives it. That would leave nothing for the estate and more to the point, nothing more for Master Copperfield. Another, perhaps more cautious, way to increase the income is to sign over the entire £1000 to a Mutual Society in return for what is termed an *annuity*. This would provide a guaranteed income for the whole of Master Copperfield's lifetime.

It should be appreciated however that unless special terms have been negotiated at the outset, annuity income generally dies with the recipient leaving nothing for the heirs'

David Copperfield nodded as he leaned back in his chair. He said 'I must admit that the notion of slowly running down the fund by drawing a trifle more income over and above the £40 interest has already occurred to me. Unfortunately I encountered some arithmetical difficulties when attempting to calculate the number of years the fund would last for the various hypothetical amounts of income. Ideally of course I would like to arrange matters so that the fund and myself expire together for if the fund expires first then I doubt I can be long after. It seems that on average a person of my age can expect to live for about 22 more years and I certainly wouldn't wish the last 5 or 10 of them to be passed in the workhouse as Mr. Micawber has speculated. Perhaps Heep, you or Mr. Wickfield may have some further advice to help me out of this difficulty?'

Misery Postponed

Uriah Heep held his hands clasped together in front of his chest and raised up his eyes before saying *'How much have I to be thankful for in living all these years with Mr. Wickfield! It is Mr. Wickfield's kindly intention that gave me my articles, which would otherwise not have been within the umble means of mother and self!* Indeed it was also Mr. Wickfield as kindly showed me the way to calculate the number of years a fund might last depending on the income taken from it. Mr. Wickfield effects it by means of a formula that I now propose to write on the blackboard, if you will only be so kind as to permit me?'

Hearing no objections raised he walked to the blackboard on the counting house wall and wrote there in bright yellow chalk

$$T \approx \frac{100}{R} \ln\left(\frac{\varepsilon}{\varepsilon - 1}\right). \qquad (1)$$

David Copperfield stared with bafflement at the simple rule written so neatly on the dark counting house blackboard and he fell speechless for a minute or two. At last he broke the silence with "You may say, Heep, that it is 'easily shown' but I must admit it's not so easy for me. I assume that T is the number of years that the fund will last and most probably the symbol R signifies the interest rate as the number of percent per annum, or 4 in this case? However I'm at a loss to know what the symbol ε might be and for the life of me I can't even guess how 'ln' is pronounced let alone what that stands for, if anything. Some additional explanation would be welcome please Heep."

'Bless you kindly Sir' said Heep, 'Why it's simplicity itself. "ln" is pronounced "l" "n" and stands for nothing more nor less than the natural logarithms listed in this book of tables. ε we might term the "income multiple" for want of a better description and is defined to be the ratio of the annual income taken at the end of each year to the interest earned in the first year.

In the calculations it is assumed that after the initial investment of £1000 no further debits or credit transactions are allowed on the account, except that at the end of every year the interest for that year is credited to the account and the income is paid out. The income is considered fixed in that the same absolute amount of income will be taken at the end of every year. The interest earned for the year, however, will vary and each year it will be R % of whatever the balance was throughout the year. At 4% per annum the interest earned for the first year would be £40 and if we decided to draw out more income than the interest earned, let us say for example £50 every year, then the income multiple would be $\varepsilon = \frac{\text{£}50}{\text{£}40}$ or 1.25. This means that immediately after the end of the first year the account balance would be £1000 + £40 interest - £50 income = £990. We can see that by taking such a large income we are effectively spending some of the capital. The interest for the next year would be 4% of £990 or £39 12s 0d. The amount of interest earned in subsequent years will be even less and will continue to decrease as the balance remaining to earn further interest is correspondingly reduced. As time passes therefore each £50 income payment would have to be made up of a growing proportion of capital and a shrinking proportion of interest.

Let us now continue with the case when £50 annual income is taken; substituting the value $\varepsilon = 1.25$ into formula (1) we obtain $T \approx \frac{100}{4} \ln\left(\frac{1.25}{0.25}\right)$. In round numbers sir it means the annual income payments could be maintained for a useful 40 years.'

'Marvellous and crystal clear' said Copperfield. 'This means that for my £1000 I can take £50 income every year for 40 years! That is much more like it and my congratulations to Mr. Wickfield on his handsome little formula.'

'I should point out ' said Heep 'that formula (1) is only an approximation to the true position, albeit a good approximation, and generally good enough for most practical purposes. However if a more accurate figure should be required it can be obtained using a version free of approximation'. Another formula now joined the first on the blackboard:

$$T = \frac{\ln\left(\frac{\varepsilon}{\varepsilon - 1}\right)}{\ln\left(1 + \frac{R}{100}\right)} \tag{2}$$

' This more accurate version shows that the £50 income can actually be taken for 41.0 years, slightly longer than the 40 years previously calculated.'

Heep continued on. 'With Micawber's permission we might now analyse the situation referred to earlier where £40 0s 6d is the income is taken each year from the account. When compared to the £40 interest earned in the first year we find that $\varepsilon = \frac{\text{£}40\ 0\text{s}\ 6\text{d}}{\text{£}40\ 0\text{s}\ 0\text{d}} = \frac{1601}{1600}$ and the approximate formula (1) tells us that

$T \approx \frac{100}{4}\ln(1601)$. This is about 184 years(or 188.1 years using the more accurate formula). As Micawber has foreseen, misery will inevitably follow somewhere around Master Copperfield's 240th birthday.' At this sly dig Mr. Micawber cast an irritable glance in Heep's direction but refrained from comment.

Heep went on 'The new Babbage Mark IV can be easily programmed to produce a table showing how the value of *T* reduces as the income multiple is increased.' (First three columns of Table 1)

Income (£ p.a.)	Income multiple $\varepsilon = \frac{\text{income}}{£40}$	*T* (years) Flat income not linked to inflation	*T* (years) Income payments increased in line with inflation at 2.5 % p.a.
£40 0s 6d	1601/1600	188.1	32.3
£ 50	1.25	41.0	24.6
£ 60	1.50	28.0	19.8
£ 70	1.75	21.6	16.6
£ 80	2.00	17.7	14.3
£ 100	2.50	13.0	11.2

Table 1
R = 4 % p.a., The £1000 starting capital is exhausted in *T* years

Great Expectations

David Copperfield relaxed back into his chair and studied the table handed to him by Heep. Thank you kindly Heep. Clearly I am a little better set than I thought, providing always of course that I don't break any longevity records. In short it seems that with interest rates fixed at 4% per annum I could take £40 a year in perpetuity or £40 0s 6d for more than 180 years, or £50 for 41 years or even £80 for nearly 18 years. Your formula is clearly very useful for finding *T* when we are given ε but might it not also be useful to have a formula the other way round as it were so that I can choose *T* and use it to calculate ε?

Heep considered the request and pointed to the formulae on the wall saying 'The two formulae you see for *T* can both be easily rearranged to give formulae for ε. Formula (1) rearranges to give

$$\varepsilon \approx \frac{1}{1-e^{-RT/100}} \qquad (3)$$

and formula (2) tells us that

$$\varepsilon = \frac{1}{1-\left(1+\frac{R}{100}\right)^{-T}}.\text{'} \qquad (4)$$

Heep wrote up these two formulae alongside their counterparts on the board. He then walked over to Copperfield and handed him the sheet of paper he had taken from his desk drawer, saying 'I have taken the liberty of drawing up a chart using the more accurate formula (4). The chart allows the desired value of T to be selected and the value of ε can then be read off for the given value of R. For example if you want the fund to last for T = 25 years at 4% per annum then the chart (better still, use formula (4) directly) shows that ε = 1.60 and the corresponding income would thus be £64 per annum.

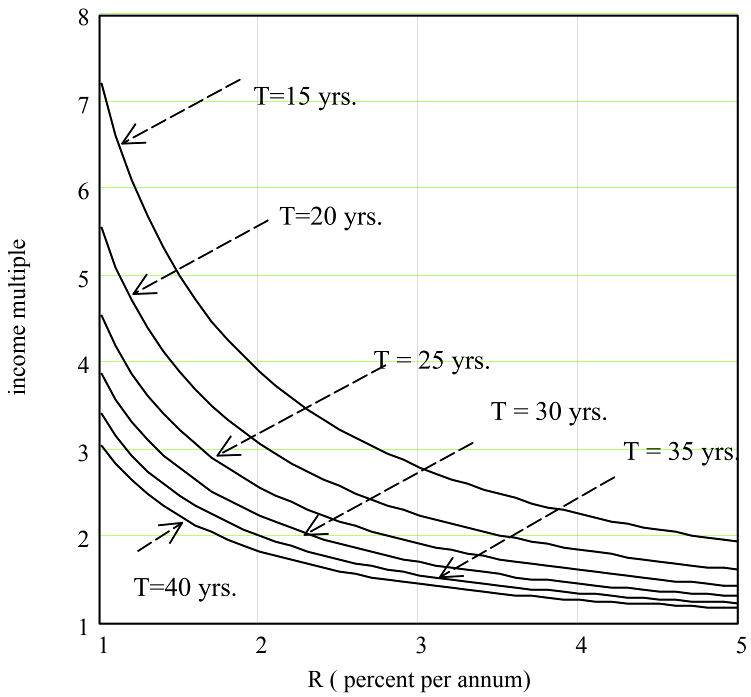

Figure 1
A chart to show how the income multiple varies with R for some values of T

'That's very useful Heep and much appreciated' said Copperfield, slipping the chart into his pocket. 'There is however still one small fly in the ointment. It would be most inconvenient for me to have to wait a whole year before picking up any income at all. How would I manage in the meantime? I would much prefer to spread the income out over the year by receiving it more frequently, say in monthly or even weekly instalments. What would become of your marvellous formulae then I wonder?'

'Why, that would present no insurmountable difficulty Master Copperfield' replied Heep. 'In fact a simple calculation leads to yet another expression that will prove useful.' Another formula joined the others on the blackboard:

$$\frac{P_k}{P_0} = (1+r)^k - \varepsilon\left[(1+r)^k - 1\right] \qquad (5)$$

Heep continued 'This formula (5) shows how the successive periodic account balances, P_k, will reduce over time if the annual income is paid out in N equal instalments over the year instead of just a single payment at the end of the year. In (5) $r = \frac{R}{100N}$ may be termed the periodic interest rate or the interest rate per interest period and the integer $k = 0,1,2......$ is used to label the interest periods with P_0 representing the £1000 starting capital. Assume now that at the end of each period we take a fixed amount of income equal to ε times the interest earned in the first period but now each period does not have to be a whole year, it can be a month or a week or whatever we choose. The amount of each income instalment will be $\varepsilon \frac{P_0 R}{100N}$ and there will be N such payments spread equally over each year. The interest is paid in and the income is taken out at the end of each period. By making ε a bigger multiple we could obtain a bigger income but run out of money sooner.

We can use equation (5) to find out how many payments can be made before the fund is completely exhausted. We see that P_k becomes zero when

$$k = \frac{\ln\left(\frac{\varepsilon}{\varepsilon - 1}\right)}{\ln\left(1 + \frac{R}{100N}\right)} \qquad (6)$$

so that with N payments per year the fund will last for T years, where now

$$T = \frac{k}{N} = \frac{\ln\left(\frac{\varepsilon}{\varepsilon - 1}\right)}{N \ln\left(1 + \frac{R}{100N}\right)} \qquad (7)$$

By setting $N = 1$ in (7) we recover our earlier situation where there was just one payment a year and obtain equation number (2) as expected.

If on the other hand in (5) we increase N without limit corresponding to the continuous compounding of interest and the continuous payment of income we find that after t years the amount of Principal remaining, $P(t)$ may now be found using

$$\frac{P(t)}{P_0} = e^{Rt/100} - \varepsilon\left[e^{Rt/100} - 1\right] \qquad (8)$$

Equation (8) may be used to determine the length of time, T, before the fund becomes exhausted (i.e. set $P(t) = 0$) and we find that T comes out to be the same as our old friend that was written in (1).'

Hard Times

David Copperfield leaned back in the chair looking interested and suitably impressed. 'I've just had a disturbing thought gentlemen. Inflation! Would it not be sensible to increase the income as time passes to make some allowance for inflation? Is there any formula to deal with that?

Uriah Heep smiled as he said 'An excellent strategy Sir, and a very prudent course of action if I may say so. To allow for inflation we can make use of formula (9) instead of formula (7). First we choose the amount of income that we wish to take as our first income payment and use it to calculate the value of ε exactly as before. Then we assume an appropriate (constant) inflation rate, let this be I % per annum. Finally we substitute our value of ε together with the assumed values for R, N and I into formula (9). After the first income payment all subsequent income payments are increased to rise in line with the assumed inflation and formula (9) tells us how long the funds will last.

$$T = \frac{k}{N} = \frac{\ln\left(\dfrac{R\varepsilon}{I + R(\varepsilon - 1)}\right)}{N \ln\left(1 + \dfrac{R - I}{I + 100N}\right)} \tag{9}$$

Formula (9) is really quite similar to formula (7). In fact (9) can actually be obtained from (7): it is as if we have used in (7) an 'effective' income multiple $\varepsilon' = \dfrac{R}{R - I}\varepsilon$ instead of ε and an 'effective' interest rate $R' = \dfrac{R - I}{1 + \dfrac{I}{100N}}$ instead of R.

Inflation has a very detrimental effect on the situation. We have already seen that when no allowance is made for inflation, R = 4% p.a. and with annual payments (N = 1) the fund provides a flat annual income of £50 a year for 41 years (ε = 1.25). If we want to increase this same £50 income in line with inflation at the rate of I = 2.5% per annum formula (9) predicts that the inflation linked equivalent income can only be supported for T = 24 years before the money is all gone.' (Some further examples are given in column 4 of table 1.)

Our Mutual Friend

Mr. Micawber had been listening to the proceedings with increasing agitation. He now jumped to his feet and said 'My dear Copperfield[2], *I am older than you; a man of some experience in life, and - and of some experience, in short, in difficulties, generally speaking. At present, and until something turns up (which I am, I may say, hourly expecting), I have nothing to bestow but advice. Still my advice is so far worth taking, that - in short, that I have never taken it myself, and am the' - here Mr. Micawber, who had been beaming and smiling, all over his head and face, up to the present moment, checked himself and frowned - 'the miserable wretch you behold.'* 'To continue Sir, my advice in short is to avoid such a hazardous strategy that in all probability could lead you into poverty and a miserable old age. A man in your prime physical condition could easily outlive his fortune if it is consumed at such rapid rates as have been mooted.

Were Mr. Wickfield himself to be present at this meeting I am convinced he would strongly encourage us to look to an annuity as the best solution. It is possible I may be of some assistance in that regard. As you know *I have entered into arrangements, by virtue of which I stand pledged and contracted to our friend Heep, to assist and serve him in the capacity of - and to be - his confidential clerk.* It is in that capacity that I have recently had occasion to investigate the subject of annuities, bringing to bear *such address and intelligence as I chance to possess.* As a result I have managed to turn up a little formula of my own that may be of interest. Taking into account the 'bonus' of mortality cross-subsidy, whereby the people who die early after taking out an annuity subsidise their fellow annuitants who live longer, my formula estimates the annual income, $I(x)$ to be expected when a Principal sum P_0 (£1000 in this case) is used to purchase a continuous life annuity by a person aged 'x' years at its commencement. I stress that such income of course would be **guaranteed** for the lifetime of the annuitant.' Mr. Micawber took the chalk in his hand and with a firm hand wrote on the board in green chalk

$$I(x) = \frac{P_0 L(x)}{\int_0^\infty L(x+t)e^{-\rho t}dt} \tag{10}$$

This was followed by a stunned silence from David Copperfield who eventually said 'Steady on please Mr. Micawber, have some pity. Now I'm completely floored and I no longer know if I'm coming or going. What is all this about and what am I supposed to make of these new hieroglyphs and what can possibly be the link between my pension and that great green snake you have drawn on the board?'

Mr. Micawber beamed and smiled. 'My dear Copperfield, one thing you could make of it indeed is that a male aged 57 taking out an annuity with £1000 would expect to receive no less than £72 9s 9d income every year for however long he lives. That very same income with Mr. Heep's method would completely exhaust the £1000 fund in just over 20 years so with an annuity you would be 'ahead' as it were if you live beyond the age of 77 years.

As to the symbols used; the function $L(x)$ is the survival function obtained using the life expectancy tables[3] for males. Out of 100,000 live births $L(x)$ is the number surviving to age x years. For instance my mortality tables tell me that for males $L(57) = 90705$, this being the number of males per 100000 live births expected to be still alive on their 57th birthday. The symbol $\rho = \frac{R}{100}$ and the time t is measured in years. As far as inflation goes one must bear in mind that nobody knows what the future rates of inflation will be and so it is impossible for any company with finite resources to guarantee an annuity income that will rise in line with inflation. However we can approximately take into account an assumed steady inflation rate at I % per annum by simply using the effective interest rate $\rho' = \frac{R-I}{100}$ in place of ρ in formula (10). I have evaluated the integral (10) by means of a cubic spline fitted to the data and the results are here in this table', he said, handing Table 2 to David Copperfield.

Age when taking up the annuity (x years)	Flat annuity income (£ p.a.) not rising with inflation	Number of years, T, the flat annuity income could be maintained using Heep's method	Annuity income payments (£ p.a.) increased in line with inflation at 2.5 % p.a. .
50	£ 62	26	£ 45
55	£ 69	22	£ 52
57	£ 72	20	£ 55
60	£ 78	18	£ 61
65	£ 91	15	£ 74
70	£ 110	12	£ 92
75	£ 137	9	£ 118

Table 2

Annuity income (ρ = 4 % p.a.) for a £1000 annuity, with and without allowance for inflation (Figures rounded to the nearest £ and nearest year)

Copperfield took Table 2 into his hand and studied it. 'This all looks very interesting, Mr. Micawber and many thanks to you for all your hard work and for sharing this with me. It certainly seems from these figures that mortality cross-subsidy is not to be sneezed at'

Rising to his feet Copperfield took one more look at the formulae on the board and said 'Well gentlemen, thank you again for your most interesting contributions. You have certainly given me plenty to think about and I shall need some time to do so. I think the rain has stopped so I will bid you good day and contact you again at a later date.'

Uriah Heep escorted David Copperfield to the counting house door. At the door and out of the earshot of Mr. Micawber he said 'You do realise Master Copperfield I hope that there's really no necessity for you to draw down your fund or to heed Micawber's call to purchase an annuity if you don't wish it? There are so many other interesting opportunities that we can talk about on your next visit. Why there are Precipice Bonds, Ostrich Farms, Time-Shares and Shipping Containers to name just a few of them so please don't hesitate to let me guide you through all these important investment decisions. You know full well you can always trust your sincere and umble servant Uriah Heep and rely on him to show you the right path. After all, if you can't trust your I.F.A., just who can you trust eh?

References

1. "David Copperfield" by Charles Dickens, Chapter 12. (Where the income is £1 and not £40)

2. A few sentences have been shamelessly purloined verbatim from Reference 1 above. Some, but not all, are italicized but in any case the reader will no doubt identify them by their superior construction.

3. http://www.gad.gov.uk/Life_Tables/Interim_life_tables.htm
Interim Life Tables, 1999 - 2001 United Kingdom Males
(x , $L(x)$)

(50, 94283), (51, 93887), (52, 93475), (53, 93029), (54, 92534),
(55, 91987), (56, 91385), (57, 90705), (58, 89946), (59, 89137),
(60, 88239), (61, 87239), (62, 86161), (63, 84989), (64, 83725),
(65, 82377), (66, 80865), (67, 79216), (68, 77410), (69, 75493),
(70, 73386), (71, 71117), (72, 68661), (73, 66025), (74, 63206),
(75, 60234), (76, 57107), (77, 53871), (78, 50489), (79, 47006),
(80, 43489), (81, 40002), (82, 36488), (83, 32913), (84, 29258),
(85, 25714), (86, 22291), (87, 19114), (88, 16160), (89, 13387),
(90, 10917), (91, 8784) , (92, 6975), (93, 5390), (94, 4074),
(95, 3008), (96, 2162), (97, 1523), (98, 1022), (99, 674),
(100, 435).

For the purposes of integration in (10) the survival function $L(x)$ was approximated by a cubic spline fitted to the above data for $50 \le x \le 100$. Visual inspection of the extrapolated spline beyond x = 100 revealed that it continued its downward descent to hit zero at about x = 102. In the absence of other data (and with the greatest respects to all 102+ year-olds) and without any justification apart from convenience the survival function has been approximated by the fitted spline in the extrapolated domain $50 \le x \le 102$ and taken to be zero for $x > 102$.