UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

JOINT MATRICULATION BOARD

GENERAL CERTIFICATE OF EDUCATION

FURTHER MATHEMATICS—Paper I

ADVANCED

Wednesday 2 June 1965 2-5

Careless and untidy work will be penalized.

Answer **seven** *questions.*

1. (a) The *rth* term of a series is $r(r+1)$. Prove by induction, or otherwise, that the sum of the first n terms of the series is $\frac{1}{3}n(n+1)(n+2)$.

(b) The rth term of a series is $\dfrac{1}{r(r+2)}$

Find the sum of the first n terms of the series and deduce the sum to infinity.

2. By considering the differential coefficient of the function $f(x)=x^3-x^2+x-3$, show that the equation $f(x)=0$ has just one real root. Determine the two consecutive integers between which the root is situated.

If α, β and γ are the roots of the equation find the value of

(i) $\alpha^2+\beta^2+\gamma^2$, (ii) $\alpha^4+\beta^4+\gamma^4$.

3. (*a*) Find the modulus *r* and the argument φ of

(i) $\dfrac{(2-i)^2}{7-i}$ (ii) $(1+\cos\theta - i\sin\theta)^2$ $\left(0<\theta<\dfrac{\pi}{2}\right)$.

(*b*) The points *A, B, P* in the Argand diagram correspond to the complex numbers a, *b, z* which satisfy the relation

$$|z-a| = k|z-b|$$

where *k* is a constant. Give a geometrical interpretation of this relation, and find the form of the locus of *P* as *z* varies

4. (*a*) Evaluate

$$\int_0^a e^{-2x}\cos 2x\,dx$$

and deduce the limiting value of the integral when *a* tends to infinity through positive values.

(*b*) A surface of revolution is formed by rotating about the *x*-axis the part of the parabola $x = at^2, y = 2at$ between t = 0 and $t = \frac{3}{4}$. Find the area of the surface.

5. The straight line whose equations are

$$\frac{x-2}{-2} = \frac{y}{1} = \frac{z+1}{2}$$

meets the plane $x+2y-2z=8$ at *B,* and *A* is the point (2, 0,-1) on the line. The foot of the perpendicular from *A* to the plane is *C.* Find (i) the coordinates of *B* and *C*, (ii) the length of *AC.* Show that the sine of the acute angle between *BA* and *BC* is $\frac{4}{9}$.

The line *AC* is produced to *D* so that *AC = 2CD.* Find the coordinates of *D.*

6. A force has components *X, Y* parallel to rectangular axes *Ox* and *Oy* respectively and *(x, y)* is a point on the line of action. Show that the force is equivalent to an equal force at the origin *O* together with a couple. State the moment of the couple.

The components of three forces in the plane are given at time *t* by

$$(2P\cos\omega t, 0), (P\cos\omega t, 2P\sin\omega t) \text{ and } (3P\sin\omega t, P\cos\omega t),$$

and their lines of action pass respectively through *O* and the points (*a*, -*a*) and (-3*a*, 2*a*) where *P*, ω and *a* are constants. If the system is reduced to a force with components X', Y' at *0* and a couple *G*. find the values of X', Y' and *G*. Deduce the equation of the line of action of the resultant, and show that this line passes through a fixed point which is independent of t.

7. The power necessary to drive a car of mass 1650 lb. along a horizontal road at a steady speed of 60 ft. per sec. is 10 h.p. At a steady speed of 20 ft. per sec. the necessary power is 2 h.p. Assuming the resistance to motion at *v* ft. per sec. to be $(a + bv^2)$ lb. wt.. find the values of the constants *a* and *b.*

If the engine is turned off when the speed of the car on the road is 60 ft. per sec. and the resistance remains the same function of *v* as before, find, to the nearest foot, the distance travelled while the speed falls to 20 ft. per,sec.

(Assume that g = 32 ft. per sec^2.)

8. A particle of mass m is tied to one end of an elastic string of natural length a and modulus $2man^2$; the other end of the string is attached to a point of a fixed smooth horizontal plane. The particle is released from rest on the plane when the string is extended to a length $2a$. (i) If the subsequent motion is unresisted find the speed of the particle when the string becomes slack. (ii) If when the extension of the string is x the motion of the particle is resisted by a force $2mn$ times the speed, show that

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + 2n^2x = 0 \quad (x \geq 0)$$

Find x as a function of t, and show that the speed of the particle at the moment the string becomes slack is $na\sqrt{2}e^{3\pi/4}$.

9. A uniform solid cone has mass M, height h and base-radius r. Show by integration that its moment of inertia about a line through the vertex and parallel to the base is

$$\frac{3}{20}M(r^2 + 4h^2)$$

(It may be assumed that the moment of inertia of a thin circular disc about a diameter is
mass x $(\text{radius})^2/4$ *.)*

The cone can swing freely under gravity about a fixed smooth pivot at the vertex. If the cone is released from rest with its axis horizontal, find the vertical component of the force on the pivot when the axis is vertical.