UNIVERSITY OF NOTTINGHAM

FACULTY OF PURE SCIENCE

FIRST YEAR PART I EXAMINATION (A), 1966

ORDINARY DEGREE EXAMINATION (A), 1966

MATHEMATICS I (iia)

SATURDAY *June* 11*th* 9-45 - 12.45

1. Show that the system

$$X - Y + 2Z - \lambda T = 4 + \lambda$$
$$3X + \lambda Y + 2Z = 6$$
$$3X + 3Y - 2Z - +2\lambda T = -2\lambda$$
$$2X + Y + \lambda Z + 3\lambda T = 2 + 4\lambda$$

is inconsistent for one and only one value of λ, that it has a non-unique solution for exactly one other value of λ, and in all other cases, has a unique solution.

Find the complete solution in the non-unique case, indicating the rank and nullity of the corresponding linear map and find
the unique solution for which $T = 1$.

2. Test the following sets of vectors for linear dependence, and find a basis for the subspace spanned by these:

(i) (0, -1, 0, 3, 6,2), (2, 4,3,-1,-2,2), (1, 1,2, 1,3, 1);

(ii) (2, 1, 3, -1), (-1,1,-2,2), (1,5,0,4).
Extend the latter set to form a basis for V_4.

[*Turn over*

3. Let $\boldsymbol{A}$, $\boldsymbol{B}$ be the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{bmatrix}.$$

Prove that

$$\det \mathbf{A} \;\; \det \mathbf{B} = (a-b)(b-c)(c-a)$$

and that

$$\det \mathbf{A} \;\; \det \mathbf{B} = \begin{bmatrix} s_0 & s_1 & s_2 \\ s_2 & s_3 & s_4 \\ s_4 & s_5 & s_6 \end{bmatrix}$$

where

$$s_r = a^r + b^r + c^r \quad (\, r = 0,1,\ldots,6)$$

Factorise the last determinant as a product of linear factors.

4. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

5. Solve the differential equations

$$(a) \;\; \frac{dy}{dx} = \frac{y-x}{8y-x+7}$$

$$(b) \;\; \frac{dy}{dx}\cos x + y \sin x = 2x \sin 2x + 2x^2 \,.$$

6. Show that the substitution $y = zx^{-1/2}$ transforms the differential equation

$$4x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2 - 1)y = 4x^{3/2}e^x \sin x$$

into one with constant coefficients, and hence obtain the general solution of the given equation.

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7. Solve the equations

(*a*) $$\frac{dy}{dx}+y=y^2(\cos x-\sin x)$$

(*b*) $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=x+x^2\log|x|+x^3$$

8. State the Dirichlet conditions under which the Fourier series corresponding to a function $f(x)$ defined in $(a, a+2T)$ converges.

Expand the function $f(x)=x^2$ in a Fourier series in the range $0<x<2\pi$, and draw a diagram indicating the sum of the resulting series for all values of $x\in[-4\pi,4\pi]$.

Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

9. Assuming that the Legendre polynomial $P_n(x)$ is defined by the expansion

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)h^n \quad (|x|\le 1,|h|<1),$$

prove that

(i) $(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$,

and hence evaluate

$$\int_{-1}^{1}xP_n(x)P_{n+1}(x)dx;$$

(ii) $$xP_n'(x)-P_{n-1}'(x)=nP_n(x)$$

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